Groups

Q.1. Show that the set of all positive rational numbers forms an abelian group under the composition defined by

$$a \circ b = \frac{(ab)}{2}$$

Solution:→ Let Q^+ denote the set of all positive rational numbers to show $(Q^+, \circ)$ is a group.

(I) Closure Property:-

For every $a, b \in Q^+$; $\frac{ab}{2} \in Q^+$

$\Rightarrow Q^+$ is a closed under the composition $\circ$.

II Associativity: Let $a, b, c \in Q^+$, then

$$(a \circ b) \circ c = \left(\frac{ab}{2}\right) \circ c = \frac{[(ab)/2] \cdot c}{2} = \frac{a[(bc)/2]}{2}$$

$$= a \circ \left(\frac{bc}{2}\right) = a \circ (b \circ c)$$

III Existence of identity:-

An element e will be the identity element if $e \in Q^+$ and if

$$e \circ a = a = a \circ e, \ \forall a \in Q^+.$$

Now, $e \circ a = a$ $\frac{(ea)}{2} = a \Rightarrow \left(\frac{a}{2}\right)(e-2) = 0$

$\Rightarrow e = 2$

Since, $a \in Q^+ \Rightarrow a \neq 0$

$2 \in Q^+$ and we have $2 \circ a = \frac{(2a)}{2} = a = a \circ 2, \forall a \in Q^+$

$\Rightarrow 2$ is the identity element.

(IV) Existence of inverse: Let $a \in Q^+$, b is the inverse of a,

then we must have $b \circ a = e = 2$

$\Rightarrow \frac{(ba)}{2} = 2 \Rightarrow b = \frac{4}{a}$

Now, $a \in Q^+ \Rightarrow \frac{4}{a} \in Q^+$.

We have $(4/a) \circ a = \left(\frac{(4/a) \cdot a}{2}\right) = 2 = a \circ (4/a)$

$\Rightarrow 4/a$ is the inverse of a

$\Rightarrow$ inverse of each element of Q^+ exist.

$\Rightarrow$

21-07-2020

(V) Commutativity: Let $a, b \in Q^+$

$\Rightarrow a \circ b = (ab)/2 = (ba)/2 = b \circ a$.

Hence, $(Q^+, \circ)$ is an abelian group

proved.

Q. 2. Show that the set Z of all integers form a group with respect to binary operation $\circ$ defined by

$a \circ b = a + b + 1$, $\forall a, b \in Z$ is an abelian group

Solution: I Closure property. Let $a, b \in Z$

$\Rightarrow a+b+1 \in Z \Rightarrow a \circ b \in Z \Rightarrow Z$ is closed with respect to $\circ$.

II Associativity: If $a, b, c \in Z$, then

$(a \circ b) \circ c = (a+b+1) \circ c = (a+b+1)+c+1 = a+b+c+2$.

Also, $a \circ (b \circ c) = a \circ (b+c+1) = a + (b+c+1) + 1 = a+b+c+2$.

$\Rightarrow (a \circ b) \circ c = a \circ (b \circ c)$, $\forall a, b, c \in Z$.

(III) Existence of identity: An element $e \in Z$ will be the identity if $e \circ a = a$, $\forall a \in Z$

Now, $e \circ a = e + a + 1$

$e + a + 1 = a \Rightarrow e = -1$

Since, $-1 \in Z$ and we have for any $a \in Z$

$(-1) \circ a = \cancel{-1} + a + \cancel{1} = a$

$\Rightarrow -1$ is the identity element

(IV) Existence of inverse: If $a \in Z$, then $b \in Z$ will be the inverse of a if $b \circ a = -1$ [$\because -1$ is the identity element]

Now, $b \circ a = -1 \Rightarrow b + a + 1 = -1 \Rightarrow b = -2 - a$

Also, $a \in Z \Rightarrow -2 - a \in Z$

and $(-2-a) \circ a = (-2-a) + a + 1 = -1$, identity element

$\therefore (-2-a)$ is the inverse of a.

(V) Commutativity: ~~Since~~ Since, $a \circ b = a+b+1 = b+a+1 = b \circ a$

$\Rightarrow$ commutativity satisfied

Hence, Z is an infinite abelian group under the given composition.

proved